Journal of Global Optimization **28:** 141–152, 2004.

An Analytic Center Cutting Plane Method for Solving Semi-Infinite Variational Inequality Problems

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(Received and accepted in revised form 21 January 2003)

Abstract. We study a variational inequality problem VI(X, F) with X being defined by infinitely many inequality constraints and F being a pseudomonotone function. It is shown that such problem can be reduced to a problem of finding a feasible point in a convex set defined by infinitely many constraints. An analytic center based cutting plane algorithm is proposed for solving the reduced problem. Under proper assumptions, the proposed algorithm finds an ϵ-optimal solution in $O^*(n^2/\rho^2)$ iterations, where $O^*(\cdot)$ represents the leading order, n is the dimension of X, ε is a user-specified tolerance, and ρ is the radius of a ball contained in the ε-solution set of VI(X, F).

AMS subject classification: 90C30

Key words: Analytic centers, Cutting plane methods, Variational inequalities

1. Introduction

Let X be a nonempty subset of R^n and F be a nonzero function from R^n into itself. The finite dimensional variational inequality problem, denoted by VI(X, F), is to find a vector $x^* \in X$ such that

$$F(x^*)^T(x - x^*) \geqslant 0 \text{ for all } x \in X. \tag{1}$$

Over the past two decades, the field of finite dimensional variational inequality problems has experienced a rapid development in theory, algorithms, and applications. Good references can be found in Harker and Pang [7].

According to [7], the history of algorithms for solving finite dimensional variational inequalities is relatively short. For practical computation, most, if not all, algorithms work only when X exhibits certain special geometric structure (such as the positive orthant of R^n or a compact polyhedral set), or when X is defined by a finite number of inequalities and equalities.

In this paper we consider a setting with X being a nonempty, bounded set defined by

$$X = \{x \in R^n \mid h(x,t) \leqslant 0 \text{ for all } t \in T\}, \tag{2}$$

where T is a nonempty compact subset of R^n, $h : R^n \times R^n \to R$ is continuous on $R^n \times T$, and $h(\cdot,t)$ is convex for each t. Since there may be infinitely many inequalities involved in defining X, we call this setting a *semi-infinite variational inequality* problem, or SIVI(X,F) in short.

Notice that when F is a continuous pseudomonotone mapping (to be defined in later sections) from X to R^n, it is not difficult to prove that $x^* \in X$ solves SIVI(V,F) if and only if it solves a related "dual" problem, denoted by DSIVI(X,F):

$$F(x)^T(x - x^*) \geqslant 0 \quad \text{for all} \quad x \in X. \tag{3}$$

Also notice that solving DSIVI(X,F) is a convex feasibility problem [2, 3, 4], i.e., finding a point x^* in a convex set defined by an infinite number of linear inequalities

$$S = \{x^* \in R^n \mid F(y)^T x^* \leqslant F(y)^T y \;\text{ for all }\; y \in X\}. \tag{4}$$

Although finding a solution point in S is not easy by using general convex programming methods, this problem is specifically suitable for using cutting plane methods since the latter do not deal with all constraints at one time. Rather, a cutting plane method starts with a ready-made set that contains S, and gradually tighten the set by adding more cutting planes. As the initial set shrinks, it eventually finds a feasible (or near feasible) point of S in a finite number of iterations. This dynamic approach is very attractive to problems having infinitely many constraints.

A recent development in cutting plane methods is the analytic center based cutting plane method. It combines the feature of the newly developed interior point methods with the classical cutting plane scheme to achieve polynomial complexity in theory and quick convergence in practice. More details can be found in [5, 10, 11, 13].

The rest of this paper is organized as follows. Some basic properties of SIVI(X,F) are discussed in Section 2 for laying down the theoretical foundation. An analytic center cutting plane method in proposed in Section 3. Its convergence and related properties are addressed in Section 4.

2. Preliminaries

Since the analytic centers are defined to stay in the interior of a region, we make the following "interior assumption" throughout this paper.

INTERIOR ASSUMPTION: There exists an $\hat{x} \in R^n$ such that $h(\hat{x},t) < 0$ for all $t \in T$.

The interior assumption assures that X has a nonempty interior. When $h(\cdot,t)$ is convex for every t, it follows from the above assumption that X is a convex set of dimension n (a *convex body* in the terminology of convex analysis [17]). Moreover, the continuity of h on $R^n \times T$ implies that X is a closed set. Remember that in our setting, we assume that X is bounded. Consequently, X is a nonempty, convex and compact subset of R^n which leads to the following result [7, 8]:

PROPOSITION 1 *In our setting with the interior assumption, if F is a continuous mapping from X to R^n, then there exists a solution to SIVI(X, F).*

Recall some definitions from [14].

DEFINITION 2 A mapping F is said to be

(i) pseudomonotone on X, if for every pair of distinct points $x, y \in X$,

$$F(x)^T(y - x) \geqslant 0 \text{ implies } F(y)^T(y - x) \geqslant 0. \tag{5}$$

(ii) pseudo-co-coercive with modulus α on X, if for any pair of distinct points $x, y \in X$,

$$F(x)^T(y - x) \geqslant 0 \text{ implies } F(y)^T(y - x) \geqslant \alpha\|F(x) - F(y)\|^2. \tag{6}$$

Directly from this definition we see that a monotone mapping is pseudomonotone and a pseudo-co-coercive mapping is also pseudomonotone. Next we show that SIVI(X, F) and DSIVI(X, F) are equivalent under pseudomonotonicity.

PROPOSITION 3 *Let F be a continuous pseudomonotone mapping over X. In our setting with the interior assumption, $x^* \in X$ is a solution to SIVI(X, F) if and only if it solves DSIVI(X, F).*

Proof. If $x^* \in X$ solves SIVI(V, F), then

$$F(x^*)^T(x - x^*) \geqslant 0 \text{ for all } x \in X.$$

From the pseudomonotonicity of F, we have

$$F(x)^T(x - x^*) \geqslant 0 \text{ for all } x \in X.$$

Conversely, let $x^* \in X$ be a solution of DSIVI(X, F). For any $x \in X$ and $x \neq x^*$, let $x_t = (1 - t)x^* + tx, t \in (0, 1)$. Then $x_t \to x^*$ as $t \to 0$. From (3)

$$F(x_t)^T(x_t - x^*) = F(x_t)^T(t(x - x^*)) = tF(x_t)^T(x - x^*) \geqslant 0.$$

Therefore $F(x_t)^T(x - x^*) \geqslant 0$. Since F is continuous at x^*, we have $F(x^*)^T(x - x^*) \geqslant 0$. □

Note in the proof that any solution to DSIVI(X, F) is a solution to SIVI(X, F) even without pseudomonotonicity.

Associated with SIVI(X, F), a gap function $g(x)$ can be defined as follows.

DEFINITION 4 Let SIVI(X, F) be given. The primal gap function is

$$g(x) = \max_{y \in X} F(x)^T(x - y) \text{ for each } x \in X. \tag{7}$$

Note that $g(x) \geqslant 0$ for all $x \in X$, and $g(x^*) = 0$ if and only if x^* is a solution to SIVI(X, F). In general, the gap function g is nonconvex and nonsmooth. However, in our setting the value of g can be computed by using some semi-infinite programming algorithms [1].

For any given number $\varepsilon > 0$, the concept of "near-optimal solutions" can be defined as follows.

DEFINITION 5 A point $\bar{x} \in X$ is called an ε-solution of SIVI(X, F) if $g(\bar{x}) \leqslant \varepsilon$.

Let S_ε be the set of all ε-solutions. Obviously, S_ε is compact. We next show it contains a small n-dimensional ball.

In our setting, since X is compact, there exists a constant d>0 such that $\|x - y\| \leqslant d$ for any $x, y \in X$. Moreover, when F is continuous, F must be uniformly bounded over X. We let $M > 0$ be an upper bound on the norm of F over X.

PROPOSITION 6 *Assume that F is Lipschitz continuous on X with modulus L. Let x^* be any solution to SIVI(X, F) and $B(x^*, \delta)$ be a ball with center at x^* and radius $\delta = \varepsilon/(M + Ld)$. Then every point $\tilde{x} \in B(x^*, \delta) \cap X$ is an ε-solution of SIVI(X, F).*

Proof. From the assumptions, for every $y \in X$ and $\tilde{x} \in B(x^*, \delta) \cap X$, we have

$$\begin{aligned} F(\tilde{x})^T(\tilde{x} - y) &= F(\tilde{x})^T(\tilde{x} - x^*) + F(\tilde{x})^T(x^* - y) \\ &\leqslant F(\tilde{x})^T(\tilde{x} - x^*) + (F(\tilde{x}) - F(x^*))^T(x^* - y) \\ &\leqslant M\|\tilde{x} - x^*\| + L\|\tilde{x} - x^*\|d \\ &\leqslant (M + Ld)\delta = \varepsilon \end{aligned}$$

Thus $\tilde{x}$ is an ε-solution of SIVI(X, F). □

From Proposition 6 it is clear that S_ε contains the convex body $B(x^*, \delta) \cap X$, therefore S_ε contains an n-dimensional ball $B(x', \rho(\varepsilon))$, where $x' \in B(x^*, \delta) \cap X$ and $\rho(\varepsilon)$ is a small positive number dependent on ε. This will guarantee the finite termination of an analytic cutting plane method.

3. Analytic center cuts for finding a near-optimal solution

Goffin et al. [2, 3] proposed an analytic center based cutting plane method for convex and quasiconvex feasibility problems and used it for solving pseudomonotone variational inequalities with X being a polyhedron [4]. The method dealt with the problem of finding an interior point solution of a convex body $\bar{S}$ defined by an oracle that for every $\bar{w}$ either returns that $\bar{w} \in \bar{S}$ or generates a separating hyperplane (a "cut") $\{w \in R^n \mid a^T w \leqslant a^T \bar{w}\} \supset \bar{S}$, with $\|a\| = 1$ being assumed for the normal vector a. Let

$$A^k = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_k^T \end{bmatrix}$$

be the matrix and c^k be the right hand side vector generated by the first k cuts. We represent $\Omega^k = \{w \in R^n \mid A^k w \leqslant c^k\}$. Then the analytic center of Ω^k is the unique point which maximizes the potential function $\sum_{j=1}^{k} \log(c_j - a_j^T w)$ on $\{w \in R^n \mid c^k - A^k w > 0\}$.

To make their method work, Goffin et al. [3] made the following general assumptions:

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(i) The solution set $\bar{S}$ is contained in the cube $\Omega^0 = \{w \in R^n \mid 0 \leqslant w \leqslant e\}$, where e is a vector of all ones.

(ii) The set $\bar{S}$ contains a full dimensional ball with radius ρ.

Note that the first part of this assumption can be met by re-scaling the problem. The second part is more essential since the radius ρ appears in the complexity estimation. Their method works as follows.

step 0. Let $A^0 = \begin{bmatrix} I \\ -I \end{bmatrix} \in R^{2n \times n}$, $c^0 = \begin{bmatrix} e \\ 0 \end{bmatrix}$, and $k = 0$.

step 1. Compute an (approximate) analytic center w^k of $\Omega^k = \{w \in R^n \mid A^k w \leqslant c^k\}$.

step 2. Check the oracle to see if $w^k \in \bar{S}$ or not. If yes, stop.

step 3. (cut generation) Generate a hyperplane

$$\{w \mid a_{k+1}^T w \leqslant a_{k+1}^T w^k\} \supset \bar{S} \quad \text{with} \quad \|a_{k+1}\| = 1. \tag{8}$$

Set

$$\Omega^{k+1} = \{w \in R^n \mid A^{k+1} w \leqslant c^{k+1}\},$$

where

$$A^{k+1} = \begin{bmatrix} A^k \\ a_{k+1}^T \end{bmatrix} \quad \text{and} \quad c^{k+1} = \begin{bmatrix} c^k \\ a_{k+1}^T w^k \end{bmatrix}.$$

step 4. Set k to $k+1$ and go to step 1.

Under the above mentioned assumptions, Goffin et al. [3] showed in the next theorem that their method is a polynomial-time approximation algorithm.

THEOREM 8 *The above column generation algorithm stops as soon as the iteration number k satisfies that*

$$\frac{2n+k}{\frac{1}{2}+2n\ln(1+\frac{k}{8n^2})} \geqslant \frac{n}{\rho^2}.$$

Now we modify Goffin et al.'s approach to solve SIVI(X, F). Let ε' be a positive number smaller than ε (will be specified in Lemma 10), $\delta = \varepsilon'/M + Ld$. Consider the ball $B(x', \rho(\varepsilon'))$, centered at x' and lying in $X \cap B(x^*, \delta)$. From Proposition 6 it is clear that $B(x', \rho(\varepsilon')) \subset S_{\varepsilon'} \subset S_\varepsilon$. Following Goffin et al., we assume in this section that $B(x', \rho(\varepsilon'))$ is contained in Ω°.

Proposed Algorithm (for finding a near-optimal solution to SIVI(X, F))

step 0. Let $\varepsilon > 0$ be a user-specified tolerance, $k = 0$ be an index,

$$A^0 = \begin{bmatrix} I \\ -I \end{bmatrix} \in R^{2n\times n},$$

$$c^0 = \begin{bmatrix} e \\ 0 \end{bmatrix} \in R^{2n}, \quad x^0 = \tfrac{1}{2}e \in R^n,$$

and

$$s^0 = c^0 - A^0x^0 = \frac{1}{2}\begin{bmatrix} e \\ e \end{bmatrix} \in R^{2n}$$

step 1. (center computation) Find an (approximate) analytic center x^k of $\Omega^k = \{x \in R^n \mid A^k x \leqslant c^k\}$.

step 2. Solve $\max_{t\in T} h(x^k, t)$. Let t^k be an element in T at which the maximum is attained. If $h(x^k, t^k) \leqslant 0$, go to step 4. Otherwise, compute a subgradient $a \in \partial_x h(x^k, t^k)$ and continue.

step 3. (feasibility-cut generation) Generate the cutting plane

$$H_k = \left\{x \in R^n \mid a^T(x - x^k) = 0\right\}.$$

Set

$$A^{k+1} = \begin{bmatrix} A^k \\ \frac{a}{\|a\|} \end{bmatrix} \quad \text{and} \quad c^{k+1} = \begin{bmatrix} c^k \\ \frac{a^T x^k}{\|a\|} \end{bmatrix}.$$

Go to step 6.

step 4. (stopping criterion) If $g(x^k) \leqslant \varepsilon$, then stop and output x^k as a near optimal solution.

step 5. (optimality-cut generation) If $F(x^k) = 0$, stop and output x^k as a solution. Otherwise, generate a cutting plane

$$H_k = \left\{ x \in R^n \mid \frac{F(x^k)^T x}{\|F(x^k)\|} = \frac{F(x^k)^T x^k}{\|F(x^k)\|} \right\}.$$

Set

$$A^{k+1} = \begin{bmatrix} A^k \\ \frac{F(x^k)}{\|F(x^k)\|} \end{bmatrix} \quad \text{and} \quad c^{k+1} = \begin{bmatrix} c^k \\ \frac{F(x^k)^T x^k}{\|F(x^k)\|} \end{bmatrix}.$$

step 6. Increase k by 1 and go to step 1.

Notice that here we split the procedure to reflect two situations based on the feasibility of x^k. If x^k is not a feasible solution of SIVI(X, F), we find a t^k with $h(x^k, t^k)>0$ and then a hyperplane is constructed in step 3 using a subgradient a. This guides the iterates to be feasible to SIVI(X, F). On the other hand, in step 5, we use $F(x^k)$ to construct the hyperplane which leads the iterates to reach the optimal solution set. To guarantee the convergence of this modified algorithm, we need to verify that the set $B(x', \rho(\varepsilon'))$ is contained in Ω^k for all k.

LEMMA 9 *For any feasibility cut, we have* $a \neq 0$ *and* $X \subset \{x \in R^n \mid a^T(x - x^k) \leqslant 0\}$.

Proof. If $a = 0$ in step 3, then x^k minimizes $h(x, t^k)$ by the convexity of $h(\cdot, t^k)$. Since the feasible set X is nonempty, it follows that $h(x^k, t^k) \leqslant 0$. This is, however, a contradiction since one has $h(x^k, t^k)>0$ in step 2.

Note that $h(x, t^k) \leqslant 0$ and $h(x^k, t^k)>0$ for any $x \in X$. By convexity we know that

$$a^T(x - x^k) \leqslant h(x, t^k) - h(x^k, t^k) < 0,$$

which implies that $X \subset \{x \in R^n \mid a^T(x - x^k) \leqslant 0\}$. □

We need to put some stronger conditions on the mapping F to show $B(x', \rho(\varepsilon')) \subset \Omega^k$.

LEMMA 10 *Given SIVI*(X, F), *assume that F is Lipschitz continuous on X with modulus L and pseudo-co-coercive on X with modulus* α. *If the proposed algorithm does not stop in steps* 4 *or* 5 *in the first k iterations, then we have* $B(x', \rho(\varepsilon')) \subset \Omega^k$ *where* $\varepsilon' + d\sqrt{\frac{\varepsilon'}{\alpha}} \leqslant \varepsilon$.

Proof. From Lemma 9, we know that for adding any feasible cut H_j. We have X remained as a subset of $\{x \in R^n \mid a^T(x - x^j) \leqslant 0\}$. Therefore, if $B(x', \rho(\varepsilon')) \not\subset$

Ω^k, there must exist an index i, $1 \leqslant i \leqslant k$, that associates with an optimality cutting plane $H_i = \{x \in R^n \mid F(x^i)^T(x - x^i) = 0\}$. There also has a point $\bar{x}^i \in B(x', \rho(\varepsilon'))$ such that $F(x^i)^T(\bar{x}^i - x^i) > 0$. Since the solution x^* of SIVI(X, F) always satisfies $F(x^i)^T(x^* - x^i) \leqslant 0$, we must have a point $\tilde{x}$ lying on the segment $[x^*, \bar{x}^i]$ and H_i. Since both x^* and $\bar{x}^i$ are in $B(x', \rho(\varepsilon'))$, $\tilde{x} \in B(x', \rho(\varepsilon'))$ and, from Proposition 6, $\tilde{x}$ is an ε'-solution to SIVI(X, F), i.e.,

$$F(\tilde{x})^T(x - \tilde{x}) \geqslant -\varepsilon' \text{ for all } x \in X.$$

However, from the pseudo-co-coercivity of F and $F(x^i)^T(\tilde{x} - x^i) = 0$, we have $\|F(x^i) - F(\tilde{x})\|^2 \leqslant \frac{1}{\alpha} F(\tilde{x})^T(\tilde{x} - x^i)$. Consequently, $\|F(x^i) - F(\tilde{x})\|^2 \leqslant \frac{\varepsilon'}{\alpha}$, and

$$\begin{aligned} F(x^i)^T(x - x^i) &= F(x^i)^T(x - \tilde{x}) + F(x^i)^T(\tilde{x} - x^i) \\ &= F(\tilde{x})^T(x - \tilde{x}) + (F(x^i) - F(\tilde{x}))^T(x - \tilde{x}) \\ &\geqslant -\varepsilon' - d\sqrt{\frac{\varepsilon'}{\alpha}} \\ &\geqslant -\varepsilon \end{aligned}$$

for any $x \in X$. This means that x^i is an ε-solution to SIVI(X, F) and causes a contradiction. □

Using Lemma 10 and following Goffin et al.'s logic in Theorem 8, we can derive a similar result that can be used as a stopping criterion for the proposed algorithm as follows.

THEOREM 11 *In the kth iteration of the proposed algorithm, if $B(x', \rho(\varepsilon')) \subset \Omega^k$, then*

$$\frac{n}{2n + k}\left(\frac{1}{2} + 2n\ln\left(1 + \frac{k}{8n^2}\right)\right) \geqslant \rho(\varepsilon')^2. \tag{9}$$

Consequently, we have the following result:

THEOREM 12 *Given SIVI(X, F) with F being nonzero Lipschitz continuous on X with modulus L and pseudo-co-coercive on X with modulus α, the proposed algorithm terminates with an ε-solution in at most $k = O^*(n^2/\rho(\varepsilon')^2)$ iterations, where O^* means the leading order in the sense of $k/\ln k = O(n^2/\rho(\varepsilon')^2)$.*

Proof. It can be readily seen that the inequality (9) will be violated as soon as

$$\frac{n}{2n + k}\left[\frac{1}{2} + 2n\ln(8n^2 + k) - 2n\ln(8n^2)\right] < \rho(\varepsilon')^2. \tag{10}$$

A sufficient condition for (10) to be valid is

$$\frac{\ln(8n^2 + k)}{2n + k} < \frac{\rho(\varepsilon')^2}{2n^2},$$

which in turn can be implied by requiring $k > 2n$ such that

$$\frac{\ln k}{k} \leqslant \frac{\rho(\varepsilon')^2}{4n^2}. \tag{11}$$

Thus, the algorithm will terminate in $O^*(n^2/\rho(\varepsilon')^2)$ iterations. □

4. Pseudomonotone semi-infinite variational inequality problems

In this section we show that the proposed analytic center cutting plane method also applies for F being pseudomonotone plus on X.

DEFINITION 13 A mapping $F : R^n \to R^n$ is pseudomonotone plus on X if it is pseudomonotone on X and, for all $x, y \in X$, if $F(x)^T(y-x) \geqslant 0$ and $F(y)^T(y-x) = 0$ then $\|F(y) - F(x)\| = 0$.

LEMMA 14 *Assume that SIVI(X, F) is given with F being continuous and pseudomonotone plus on X. When the proposed algorithm is applied, if none of the analytic centers x^j of Ω^j, for $j = 1, 2, \ldots$, is optimal to SIVI(X, F), then H_j contains no solution of SIVI(X, F).*

Proof. Let x^* be a solution to SIVI(X, F). One possibility is that $x^* \in H_j$ for some j with the analytic center x^j being feasible to SIVI(X, F). In this case we have $F(x^j) = F(x^*)$, since F is pseudomonotone plus and $F(x^j)^T(x^* - x^j) = 0$. Consequently, for any $x \in X$, we have

$$\begin{aligned} F(x^j)^T(x - x^j) &= F(x^j)^T(x - x^*) + F(x^j)^T(x^* - x^j) \\ &= F(x^*)^T(x - x^*) \\ &\geqslant 0. \end{aligned}$$

This implies that x^j is an optimal solution of SIVI(X, F) and causes a contradiction.

The other possibility is that $x^* \in H_j$ for some j with the analytic center x^j being infeasible to SIVI(X, F). In this case we have $a^T(x^* - x^j) = 0$ and $h(x^j, t^j) \leqslant h(x^*, t^j) \leqslant 0$. This again causes a contradiction to the process of constructing feasibility cuts. □

THEOREM 15 *Assume that SIVI(X, F) is given with F being nonzero, Lipschitz continuous on X with modulus L and pseudomonotone plus on X. The proposed algorithm either terminates with an ε-solution of SIVI(X, F) or generates a subsequence of the analytic centers $\{x^k\}$ converging to a solution of SIVI(X, F).*

Proof. Assume that the proposed algorithm does not terminate with an ε-solution of SIVI(X, F). We can construct a sequence of balls $\{B(\bar{x}_i, \varepsilon_i)\}_{i=1,2,\ldots}$ with $\bar{x}_i$ being the center and ε_i the radius satisfying the following two conditions:

(i) $B(\bar{x}_i, \varepsilon_i) \subset \text{int}(B(x^*, \delta) \cap X)$,

(ii) $\bar{x}_i \to x^*$ and $\varepsilon_i \to 0$, when $i \to \infty$,

where δ is given as in Proposition 6.

From Theorem 11, we know for each i there exists an index m_i such that $B(\bar{x}_i, \varepsilon_i) \not\subset \Omega^{m_i}$. Therefore, we know there exists a point $\hat{x}^i \in B(\bar{x}_i, \varepsilon_i)$ and a cutting plane $H_{n_i} = \{x \mid F(x^{n_i})^T(x - x^{n_i}) = 0\}$, $n_i \leqslant m_i$, such that

$$F(x^{n_i})^T(\hat{x}^i - x^{n_i}) > 0.$$

From Proposition 3 and Lemma 14, we have

$$F(x^{n_i})^T(x^* - x^{n_i}) < 0.$$

Thus we must have a point $\tilde{x}^i$ that lies on the segment $[x^*, \hat{x}^i]$ and H_{n_i}. Obviously the sequence $\{\tilde{x}^i\} \to x^*$. Since X is compact, there are two subsequences $\{x^{n_i}\}_{i\in Z}$ and $\{\tilde{x}^i\}_{i\in Z}$ ($Z \subset \{1, 2, \ldots\}$) satisfying that $\{\tilde{x}^i\}_{i\in Z} \to x^*$ and $\{x^{n_i}\}_{i\in Z} \to x^\triangle$ when $i \to \infty$. Since $x^{n_i} \in X$ for each $i \in Z$ and X is compact, we have $x^\triangle \in X$. Moreover, since

$$F(x^{n_i})^T(\tilde{x}^i - x^{n_i}) = 0,$$

from the continuity of F, by letting $i \to \infty$, we have

$$F(x^\triangle)^T(x^* - x^\triangle) = 0. \tag{12}$$

Because $F(x^*)^T(x^\triangle - x^*) \geqslant 0$ and F is pseudomonotone plus, it follows that $F(x^*) = F(x^\triangle)$. We now show that $x^\triangle$ is a solution of SIVI(X, F). For any $x \in X$, we have

$$\begin{aligned} F(x^{n_i})^T(x - x^{n_i}) &= F(x^{n_i})^T(x - \tilde{x}^i) + F(x^{n_i})^T(\tilde{x}^i - x^{n_i}) \\ &= F(x^{n_i})^T(x - \tilde{x}^i). \end{aligned}$$

Hence

$$\begin{aligned} F(x^\triangle)^T(x - x^\triangle) &= F(x^\triangle)^T(x - x^*) \\ &= F(x^*)^T(x - x^*) \\ &\geqslant 0. \end{aligned}$$

The desired result follows. □

5. Final remarks

We have proposed an analytic center cutting plane method for solving semi-infinite variational inequality problems. It is shown that the method has polynomial complexity of $O^*(n^2/\rho^2)$ under proper assumptions, where $O^*(\cdot)$ represents the leading order and $\rho = \rho(\epsilon')$. The results can be extended if nonlinear cuts, rather than linear cuts are introduced. We could also consider a multiple-cut version of the algorithm to improve the computational efficiency of the algorithm. These topics have been discussed extensively for convex feasibility problems (e.g., [6, 9, 10, 11, 12, 15, 16, 18]) but have not been studied in the semi-infinite variational inequalities setting. It is our belief that the research on the semi-infinite case is of importance in practice and we hope this paper may stimulate further investigation in this direction.

6. Acknowledgments

J. Sun's work was partially supported by the Singapore-MIT Alliance and by Grants R314-000-026/028/042-112 of National University of Singapore.

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